# Chapter 2 Solutions

* 1. Derive the discrete-time model of Example 2.1 from the solution of the system differential equation with initial time *kT* and final time(*k*+1)*T.*

The volumetric fluid balance gives the analog mathematical model



where = *R C* is the fluid time constant for the tank. The solution of this equation is



Let *qi* be constant over each sampling period *T*, i.e. *qi*(*t*) = *qi*(*k*) = constant, for *t* in the interval

[*kT,* (*k*+1)*T*). Then

(i) Let *t*0 = *kT*, *t* = (*k* + 1)*T*

(ii) Simplify the integral as follows with

 

We thus reduce the differential equation to the difference equation



* 1. For each of the following equation, determine the order of the equation then test it for

(i) Linearity. (ii) Time-invariance. (iii) Homogeneousness.

(a) *y*(k+2) = *y*(*k*+1) *y*(*k*) + *u*(*k*)

(b) *y*(*k*+3) + 2 *y*(*k*) = 0

(c) *y*(*k*+4) + *y*(*k*-1) = *u*(*k*)

(d) *y*(*k*+5) = *y*(*k*+4) + *u*(*k*+1) − *u*(*k*)

(e) *y*(k+2) = *y*(*k*) *u*(*k*)

The results are summarized below

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Problem | Order | Linear | Time-invariant | Homogeneous |
| (a) | 2 | No | Yes | No |
| (b) | 3 | Yes | Yes | Yes |
| (c) | 5 | Yes | Yes | No |
| (d) | 5 | Yes | Yes | No |
| (e) | 2 | No | Yes | No |

* 1. Find the transforms of the following sequences using Definition 2.1

(a) {0, 1, 2, 4, 0, 0,...} (b) {0, 0, 0, 1, 1, 1, 0, 0, 0,...}

(c) {0, 2−0.5 , 1, 2−0.5 , 0, 0, 0, ... }

From Definition 2.1, {*u*0, *u*1 , *u*2 , ... , *uk* , ... } transforms to . Hence:

(a)  (b) 

1. 
	1. Obtain closed forms of the transforms of Problem 2.3 using the table of z-transforms and the time delay property.

Each sequence can be written in terms of transforms of standard functions

(a) {0, 1, 2, 4,0,0,...} = {0, 1, 2, 4, 8, 16,...} − {0, 0, 0, 0, 8, 16,...}={f(k)}−{g(k)}

where  



(b) {0, 0, 0, 1, 1, 1, 0, 0,...} = {0, 0, 0, 1, 1, 1, 1, 1,...} − {0, 0, 0, 0, 0, 0, 1, 1, 1, 1,...}

= {f(k)}− {g(k)}

where  



(c) {0,2-0.5,1,2-0.5,0,0,...} = {0,2-0.5,1,2-0.5,0,-2-0.5,-1,-2-0.5,0,...}+ {0,0,0,0,2-0.5,1,2-0.5,0,-2-0.5,-1,-2-0.5,0,...}

= {f(k)} + {g(k)}

where  



* 1. Prove the linearity and time delay properties of the z-transform from basic principles.

To prove linearity, we must prove homogeneity and additivity using Definition 2.1,

(i) Homogeneity: 





(ii) Additivity 



To prove the time delay property, we write the transform of the delayed sequence



1. Use the linearity of the z-transform and the transform of the exponential function to obtain the transforms of the discrete-time functions.

(a) sin(*kT*) (b) cos(*kT*)

(a) 



(b) 



* 1. Use the multiplication by exponential property to obtain the transforms of the discrete-time functions.

(a) *e*−*kT*sin(*kT*) (b) *e*−*kT*cos(*kT*)

The multiplication by exponential property with  gives



(a) 

(b) 

* 1. Find the inverse transforms of the following functions using Definition 2.1 and, if necessary, long division

(a)  (b) 

(c)  (d) 

Use Definition 2.1 to obtain

(a)  (b) 

(c) 

 

(d) 





1. For Problems ‎2.8.(c), (d), find the inverse transforms of the functions using partial fraction expansion and table look-up.

(c) 

  

(d) (i) 

We obtain  and use the identities









 

sin(A+B) = sin(A) cos(B) + cos(A) sin(B)

(ii) Using MATLAB

>> num=[1,-0.1]; den=[1,0.04,0.25,0];

>> [r,p,k]=residue(num,den)

r =

 0.2000 - 1.0088i

 0.2000 + 1.0088i

 -0.4000 + 0.0000i

p =

 -0.0200 + 0.4996i

 -0.0200 - 0.4996i

 0.0000 + 0.0000i

>> abs(r(1))

ans =

 1.0284

>> angle(r(1)) % Should be equal to (see answer obtained earlier)

ans =

 -1.3751

>> abs(p(1))

ans =

 0.5

>> angle(p(1))

ans =

 1.6108

Use the formula

1. Use the complex differentiation property and the transform of the unit ramp to obtain the z-transform of the parabolic

Solution

The transform of the ramp is

Using the complex differentiation property, we have

1. Solve the following difference equations

(a) *y*(*k*+1) − 0.8 *y*(*k*) = 0, y(0) = 1

(b) *y*(*k*+1) − 0.8 *y*(*k*) = 1(*k*), y(0) = 0

(c) *y*(*k*+1) − 0.8 *y*(*k*) = 1(*k*), y(0) = 1

(d) *y*(*k*+2) + 0.7 *y*(*k*+1) + 0.06 *y*(*k*) = (*k*), y(0)=0, y(1)=2

(a) *y*(*k*+1) − 0.8 *y*(*k*) = 0, y(0) = 1

z-transform

 

(b) *y*(*k*+1) − 0.8 *y*(*k*) = 1(*k*), y(0) = 0

z-transform





 

(c) *y*(*k*+1) − 0.8 *y*(*k*) = 1(*k*), y(0) = 1

The solution is the sum of the solutions from (a) and (b)

 

(d) *y*(*k*+2) + 0.7 *y*(*k*+1) + 0.06 *y*(*k*) = (*k*), y(0)=0, y(1)=2

z-transform









Without dividing by we get

* 1. Find the transfer functions corresponding to the difference equations of Problem 2.2 with input *u*(*k*) and output *y*(*k*). If no transfer function is defined, explain why.

(a) and (e) are nonlinear and (b) is homogeneous. They have no transfer functions.

(c) *y*(*k*+4) + *y*(*k*−1) = *u*(*k*)

Z-transform  

(d) *y*(*k*+5) = *y*(*k*+4) + *u*(*k*+1) − *u*(*k*)

z-transform  

1. The following difference equation describes the evolution of the expected price of a commodity[[1]](#footnote-1)

where is the expected price after quarters, is the actual price after quarters, and is a constant in the range .

1. Obtain the transfer function of the system with input and output
2. Assuming a zero initial estimate, obtain the price estimate using the transfer function (i) for a fixed actual price of one unit, (ii) for an exponentially decaying price .
3. The Fibonacci sequence is a set of numbers generated by the difference equation

The sequence describes many phenomena in nature. Show that the sequence is given by

The number is known as the golden ratio. Show that it is the positive solution of the equation

Solution

z-transforming the difference equation gives

The inverse z-transform is

The characteristic equations, , has the roots

The characteristic equation, , is the same as the equation governing the golden ratio and its positive is the golden ratio.

1. Test the linearity with respect to the input of the systems for which you found transfer functions in Problem 2.12.
2. *y*(*k*+4) + *y*(*k*−1) = *u*(*k*)

The transfer function of the system is



For inputs *u*1(*k*) and *u*2(*k*), we have outputs



We now as input try the linear combination



1. *y*(*k*+5) = *y*(*k*+4) + *u*(*k*+1) − *u*(*k*)

Repeat above steps using the transfer function of (d).

1. If the rational functions of Problems ‎2.8.(c), (d), are transfer functions of LTI systems, find the difference equation governing each system.

(c) 

*y*(*k*+2 + 0.3 y(k+1) + 0.02 y(k) = *u*(*k*+1)

 (d) 

*y*(*k*+2 + 0.04 y(k+1) + 0.25 y(k) = *u*(*k*+1) − 0.1 *u*(*k*)

1. We can use *z*-transforms to find the sum of any series . This is accomplished by first recognizing that the sum

is the solution of the difference equation

where *a*(*k*) is the term in the series. Show that the z-transform of the sum is given by

then evaluate the following summations

(a)  (b) 

**Solution**

The z-transform of the difference equation is

Solving for gives the expression

1. The z-transform of the series is

The z-transform for the sum is

Inverse z-transform

1. The z-transform of the series is

The z-transform for the sum is

1. Given the discrete-time system

find the impulse response of the system :

a. From the difference equation

b. Using z-transformation

Solution

1. We consider the difference equation with the impulse input and the initial conditions Substituting in the difference equation, we have

In general, we have the impulse response

1. We z-transform the difference equation to obtain the transfer function

Inverse z-transforming gives the impulse response

The above form is identical to the one obtained in part (a) as can be verified by substituting values of .

1. The following identity provides a recursion for the cosine function

integer

To verify its validity, let and rewrite the expression as a difference equation. Show that the solution of the difference equation is indeed .

Solutions

We write the difference equation corresponding to the identity

or equivalently

We z-transform to obtain

then solve for

Substituting for the initial conditions gives

From Appendix I, we have the inverse transform

Note that the formula works for negative arguments since

1. Repeat Problem 2.19 for the identity

integer

Solutions

We write the difference equation corresponding to the identity

As in the solution of Problem 2.x, we have

Since the difference equation is identical to that of Problem 2.16, we have

Substituting for the initial conditions gives

From Appendix I, we have the inverse transform

Note that the formula works for negative arguments since the substitution gives the same identity multiplied by −1.

1. Find the impulse response functions for the systems governed by the following difference equations

(a) *y*(*k*+1) − 0.5 *y*(*k*) = *u*(*k*)

(b) *y*(*k*+2) − 0.1 *y*(*k*+1) + 0.8 *y*(*k*) = *u*(*k*)

(a) *y*(*k*+1) − 0.5 *y*(*k*) = *u*(*k*)

 

(b) *y*(*k*+2) − 0.1 *y*(*k*+1) + 0.8 *y*(*k*) = *u*(*k*)



Equating coefficients, we solve for e−*α* and *ωd* then use the tables and the delay theorem

 (zero elsewhere)

Using MATLAB

>> [l1,l2,l3]=residue(1,[1,-0.1,0.8])

l1 =

 0.0000 - 0.5599i

 0.0000 + 0.5599i

l2 =

 0.0500 + 0.8930i

 0.0500 - 0.8930i

l3 =

 []

>> abs(l1(1)), abs(l2(1))

ans =

 0.5599

ans =

 0.8944

>> angle(l1(1)), angle(l2(1))

ans =

 -1.5708

ans =

 1.5149

>> pi/2

ans =

 1.5708

(zero elsewhere)

1. Find the final value for the functions if it exists

(a)  (b) 

(a) 

(b) 

 The denominator has complex conjugate poles with magnitude  greater than unity. Therefore the corresponding time sequence is unbounded and the final value theorem does not apply.

1. Find the steady-state response of the systems due to the sinusoidal input *u*(*k*) = 0.5 sin(0.4 *k*)

(a)  (b) 

Sinusoidal input 

(a) 



*u*(*k*) =0.5 × 1.537 sin(0.4 *k* − 0.242) = 0.769 sin(0.4*k* − 0.242)

1. 



*u*(*k*) =0.5 × 0.714 sin(0.4 *k* − 0.273) = 0.357 sin(0.4 *k* − 0.273)

1. Find the frequency response of a noncausal system whose impulse response sequence is given by



**Hint:** Express the periodic impulse response sequence with period *K* as



Then Laplace transform it.

Laplace transform the sequence then let *s* = *jω*



1. The well known Shannon reconstruction theorem states that:

Any bandlimited signal *u*(*t*) with bandwidth *ωs*/2 can be exactly reconstructed from its samples at a rate *ωs* = 2*π*/*T*. The reconstruction is given by



Use the convolution theorem to justify the above expression.

By the sampling theorem, the signal can be recovered from its samples using a LPF of bandwidth *ωs*. Multiplication in the frequency domain is equivalent to convolution with the inverse transform, the sinc function in the time domain. Convolution of the samples and the sinc function yields the expression.

1. Obtain the convolution of the two sequences {1,1,1} and {1,2,3}

(a) Directly (b) Using z-transformation.

Convolution of the two sequences {f(*k*)}={1, 1, 1} and {g(*k*)}={1, 2, 3}

(a) Directly y(0) = f(0).g(0)= 11=1

y(1) = f(1).g(0) + f(0).g(1) = 11+12=3

y(2) = f(2).g(0) + f(1).g(1) + f(0).g(2) = 11+12 + 13 = 6

y(3) = f(2).g(1) + f(1).g(2) = 12 + 13 = 5

y(4) = f(2).g(2) = 13 = 3

y(*k*) = 0, *k* > 4

(b) Using z-transformation

F(*z*) = 1 + z-1 + z-2 G(*z*) = 1 + 2z-1 + 3z-2

Y(*z*) = F(*z*).G(*z*) = 1+ 3z-1 + 6 z-2 + 5z-3 + 3z-4

{y(*k*)} = {1, 3, 6, 5, 3, 0, 0, ...}

1. Obtain the modified z-transforms for the functions of Problems (2.6) and (2.7).

For 2.6-(a), 



For 2.6-(b), 



For 2.7-(a) ,



For 2.7-(b),

 

1. Using the modified z-transform, examine the intersample behavior of the functions h(k) of Problem 2.21. Use delays of (1) 0.3T, (2) 0.5T, and (3) 0.8T. Attempt to obtain the modified z-transform for Problem 2.22 and explain why it is not defined.

Solution for 2.21

For 2.21(a) 

For any value *m* 

(i) 0.3*T*, *m* = 0.7 (ii) 0.5*T*, *m* = 0.5, and (iii) 0.8*T, m* = 0.2.

For 2.21(b) 





Use the results of problem 2.21 to obtain the answer.



1. *m* = 0.7 
2. *m* = 0.5 Similarly





1. *m* = 0.2





Solution for 2.22

2.22(a) 

1. *m* = 0.7 
2. *m* = 0.5 

(iii) *m* = 0.2 

2.22(b) 



(−0.1)*m* and (−0.3)*m* are complex numbers. Thus, the sequence is not defined between sampling points.

Obtain *H*(*z*, *m*) for *m* = 0.7, 0.5, 0.2, as in (a).

1. The following open-loop systems are to be digitally feedback controlled. Select a suitable sampling period for each if the closed-loop system is to be designed for the given specifications

(a)  Time Constant = 0.1 s

(b)  Undamped natural frequency = 5 rad/s, Damping ratio = 0.7

(a) For a time constant = 0.1 s

Let

 s

(b) For rad/s, , we have rad/s,

 Let ms.

1. Repeat problem 2.29 if the systems have sensor delays of : (a) 0.025 s (b) 0.03 s

(a) *T* = 0.025 s (b) *T* = 0.03 s. (cannot sample faster than the sensor delay)

## Computer Exercises

1. Consider the closed-loop system of Problem 2.29(a)
	1. Find the impulse response of the **closed-loop** transfer function and obtain the impulse response sequence for a sampled system output.
	2. Obtain the z-transfer function by z-transforming the impulse response sequence.
	3. Using MATLAB, obtain the frequency response plots for the analog system and for sampling frequencies *ωs* = *k ωb*, *k* = 5, 35, 70.
	4. Comment on the choices of sampling periods of part (b).

The closed-loop transfer function is 

1. The impulse response is



and the impulse response sequence for a sampled system output is 

(b) The z-transform of the impulse response is



(c) The corresponding frequency response plots for sampling periods *T* = 0.1, 0.05, 0.02, 0.1 s, as well as for the analog system can be obtained using the MATLAB commands

% Exercise 2.22 Digital control text

clf

tau=0.1;% 1/wb=time constant

T=tau\*[1/5, 1/35, 1/70];

num=[10,0];

w=[.1:.05:100];

for i=1:3

den=[1,-exp(-10\*T(i))];

g=tf(num,den, T(i));

[mag,ang]= bode(g,w); % Frequency response

mm=mag(:); % Change mag to vector

plot(w,T(i)\*mm)

hold on

end

nc=1; dc=[.1, 1];

w=.1:.05:100;

[mc,ac, w]=bode(nc,dc,w); plot(w,mc,'r')



*G*(*jω*)

*ω*

**Frequency response plots for sampling frequencies *ωs* = *k ωb*, *k* = 5, 35, 70 and for the analog system for Problem 2.25.**

The frequency response plots are normalized (multiplied by *T*)to simplify their comparison. The plots for the discrete time system are closer to the analog frequency response for faster sampling. The discrete time plots are significantly different from the analog plot for *T* = 0.1 s and almost indistinguishable for *T* = 0.1/35 and 0.1/70s. This verifies the rule of thumb for the selection of the sampling rate.

1. Repeat Problem 2.31 for the second order closed-loop system of Problem 2.29(b) with plots for sampling frequencies *ωs* = *k ωd*, *k* = 5, 35, 70.

The closed-loop transfer function is 

1. The impulse response is



and the impulse response sequence for a sampled system output is



(b) The z-transform of the impulse response is



(c) The corresponding frequency response plots for sampling periods *T* = 2π/(*kωd*)s, *k*=5, 35, 70, as well as for the analog system can be obtained using the MATLAB commands

% Exercise 2\_24

clf

hold on

wn=5;zeta=0.7; % Closed-loop data

wd=wn\*sqrt(1-zeta^2); % Damped natural frequency

ttt=2\*pi/wd; T=[ttt/5,ttt/35, ttt/70]; % Sampling periods

w=[.1:.1:200];

gc=tf(wn^2,[1,2\*zeta\*wn,wn^2]); % Analog transfer function

% Plot the frequency response for the analog system

w=.1:.1:200;

[mc,ac, w]=bode(gc,w); plot(w,mc(:),'r')

% Calculate and plot discrete frequency responses

for i=1:length(T)

ti=T(i);

% numerator and denominator od z-transfer function

num=[7.0014\*exp(-3.5\*ti)\*sin(3.5707\*ti),0];

den=[1,-2\*exp(-3.5\*ti)\*cos(3.5707\*ti), exp(-7\*ti)];

g=tf(num,den,ti);

[mm,aa,w]= bode(g,w);

plot(w,ti\*mm(:))

end

(d) The frequency response show little aliasing in the frequency range of interest for *T* = 2π/(70*ωd*) s, some aliasing for *T* = 2π/(35*ωd*) s, and unacceptable aliasing *T* = 2π/(5*ωd*) s,. The analog plot (red) is similar to that of the two faster rates at low frequencies and differs from the *T* = 2π/(35*ωd*) s plot close to the folding frequency. The results confirm that the rule of thumb gives a reasonable estimate of the required sampling rate.



*G*(*jω*)

*ω*

**Frequency response plots for sampling frequencies *ωs* = *k ωb*, *k* = 5, 35, 70 and for the analog system for Problem 2.26.**

1. Use SIMULINK with a sampling period of 1s. to verify the results of Problem ‎2.23. Simulate the system for 300 s then change the axes to display the last 50 s only.

(a) 



**Simulation diagram for Problem 2.17(a) using SIMULINK.**

Problem ‎2.22(a) gives the steady-state response

*u*(*k*) = 0.769 sin(0.4*k* − 0.242)



**Sampled sinusoidal input (red) and steady-state sinusoidal (blue) for Problem 2.20(a).**

(b) 



**Simulation diagram for Problem 2.20(b) using SIMULINK.**

Problem ‎2.22(b) gives the steady-state response

*u*(*k*) = 0.357 sin(0.4 *k* − 0.273)



**Sampled sinusoidal input (red) and steady-state sinusoidal (blue) for Problem 2.20(a).**

1. Consider the model of the evolution of the expected price of a commodity of Problem 3.12

where is the expected price after quarters, is the actual price after quarters, and is a constant in the range .

1. Simulate the system with γ = 0.5 and a fixed actual price of one unit, and plot the actual and expected prices. Discuss the accuracy of the model prediction.
2. Repeat part (a) for an exponentially decaying price *p*(*k*) = (0.4)*k*.
3. Discuss the predictions of the model referring to your simulation results.

*pe*(*k*+1) = (1 − *γ*) *pe*(*k*) + *γ* *p*(*k*)

 where *pe*(*k*) is the expected price after *k* quarters, *p*(*k*) is the actual price after *k* quarters, and *γ* is a constant.

1. Simulate the system with *γ* = 0.5 and a fixed actual price of one unit and plot the actual and expected prices. Discuss the accuracy of the model prediction
2. Repeat part (a) for an exponentially decaying price *p*(*k*) = (0.4)*k*.
3. Repeat part (a) for an exponentially decaying price *p*(*k*) = (0.95)*k*.
4. Discuss the predictions of the model referring to your simulation results.

The recursion describing the solution can be easily simulated using a discrete state-space block. Although discrete state-space equations are introduced in Chapter 7, they reduce to the simple recursion of our model for the case of scalar vector x(*k*), where x(*k*) is the price *pe*(*k*). We could also avoid the use of state-space blocks by z-transforming to obtain the corresponding transfer function.

1. Simulate the system with *γ* = 0.5 and a fixed actual price of one unit and plot the actual and expected prices. Discuss the accuracy of the model prediction.



**Simulation diagram for constant price using SIMULINK.**

The model converges to the correct estimate after a few sample points. At *k* = 5, the error is less than 5%.. This is a reasonable estimate assuming that the sampling period is small relative to the time after which the price estimate is used.



**Time response of price estimator for a constant price.**

1. Repeat part (a) for an exponentially decaying price *p*(*k*) = (0.4)*k*.

We use a state space block with unity initial condition and *A*=0.4.



**Simulation diagram for exponentially decaying price using SIMULINK.**

The dynamics of the model are too slow to track the exponentially decaying price. The actual price decays much faster than the model predictions.



**Time response of price estimator for a fast exponentially decaying price.**

1. Repeat part (a) for an exponentially decaying price *p*(*k*) = (0.95)*k*.

We use a state space block with unity initial condition and *A*=0.95 and with the same simulation diagram as part (b).

The dynamics of the model are able to track the exponentially decaying price since the decay is very slow.



**Time response of price estimator for a slow exponentially decaying price.**

1. Discuss the predictions of the model referring to your simulation results.

The price estimator dynamics are able to estimate a constant price but are unable to estimate a decaying exponential if the rate of decay is fast relative to the filter dynamics. If the price decay is very slow, then the estimator is able to track the price with some error.

* 1. Write a MATLAB program that calculates the step response of the system of Example 2.21 by using the step response coefficients and formula (2.28).

SOLUTION

Hz=tf(1,[1 -0.5],1);

[g,t]=step(Hz);

n=length(g);

u=ones(1,n);

y=zeros(1,n);

y(1)=0; % u(0)=0

for k=2:n

 y(k)=y(k)+g(1)\*u(k-1);

 for i=2:k-1

 y(k)=y(k)+(g(i)-g(i-1))\*u(k-i);

 end

end

1. Gujarate, D.N., 1988. Basic Econometrics. McGraw-Hill, NY, p. 547. [↑](#footnote-ref-1)